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Instructions:

1. This exam has 8 pages. Please make sure you have all pages.
2. The point value of each problem occurs to the left of the problem.
3. **You must show correct work to receive credit.** Correct answers with inconsistent work or with no justification will not be given credit.
4. Books, notes and calculators are not allowed.
5. Turn off and put away all cell phones.

Page	Points	Points Possible
2	6	6
3	6	7
4	5	6
5	6	6
6	5	5
7	1	5
8	4	5
Total	33	40

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1. Let $\sigma \in S_8$ be given by $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 5 & 1 & 8 & 7 & 3 \end{pmatrix}$.

(a) (2 pts) Write σ as a product of disjoint cycles.

$$\sigma = (1 \ 2 \ 4 \ 5)(3 \ 6 \ 8)$$

(2)

because all the possible orbits of σ are:

$$\left(\begin{array}{c} \{1, 2, 4, 5\} \\ \{3, 6, 8\} \\ \{7\} \end{array} \right)$$

(b) (2 pts) Determine whether σ is even or odd.

no need to state (7)
in the product because
it stays same
anyway

Recall that any permutation can be written as a product of transpositions.

$$\sigma = (1 \ 2 \ 4 \ 5)(3 \ 6 \ 8)$$

(2)

$$= (1 \ 5)(1 \ 4)(1 \ 2)(3 \ 8)(3 \ 6)$$

so σ is odd since it is made from an odd number of transpositions. Of course, adding the identity which is even will keep it odd.

(c) (2 pts) Compute σ^2 .

$$\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 8 & 1 & 2 & 3 & 7 & 6 \end{pmatrix}$$

(2)

(simply because $1 \rightarrow 2 \rightarrow 4$
 $2 \rightarrow 4 \rightarrow 5$)

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2. (5 pts) Let a and c be elements of a group G . Show that if a has finite order n then cac^{-1} also has order n .

(1)

 $a, c \in G$ a has finite order $n \Leftrightarrow a^n = e$

where $n \in \mathbb{Z}$,
 n being the smallest
such number.

Note that:
the element
 $cac^{-1} \in G$

since $a \in G$
 $c \in G$

then $ca \in G$
(closure under
operation)

but $a^{-1} \in G$
since G group

$\Rightarrow a^{-1}ca \in G$
(closure under
operation).

$$(cac^{-1})^n = \underbrace{(cac^{-1}).(cac^{-1}) \dots (cac^{-1})}_{n \text{ times}}$$

since $ca, c^{-1} \in G$,
associativity holds: $= ca(c^{-1}c).a.(c^{-1}.c) \dots (c^{-1}.c).a.c^{-1}$

since G group,
 $c \in G$, $c^{-1}.c = e$ $= a \cdot e \cdot a \cdot e \dots e \cdot a \cdot c^{-1}$

since $a \cdot e = e \cdot a = a$ $\forall a \in G$ $= \underbrace{c \cdot a \cdot a \dots a}_{n \text{ times}} \cdot c^{-1}$
 $= c \cdot a^n \cdot c^{-1} = c \cdot e \cdot c^{-1}$ (since a has
finite order n)
 $= c \cdot c^{-1} = e$

$\Rightarrow cac^{-1}$ also has order n .

3. Mark each of the following true or false. Briefly justify your answers.

- (a) (2 pts) Every abelian group is cyclic.

False: A counter example is the Klein-4 group "V"
which is abelian yet not cyclic.

*	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	³ a
c	c	b	a	e

(2)

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- (b) (2 pts) There is at least one abelian group of order
- n
- for every integer
- $n > 0$
- .

True why? ①

(\mathbb{Z}_n)

- (c) (2 pts) If
- a, b, c
- are elements of a group
- G
- , the equation
- $axb = c$
- always has a unique solution in
- G
- .

~~True~~

②

$e, a, b, c \in \text{group } G$; so does all of a^{-1}, b^{-1}, c^{-1} respectively.

$$\underbrace{a \cdot u \cdot b}_{e} = c$$

$$a^{-1} \cdot a \cdot u \cdot b = a^{-1} \cdot c$$

$$e \cdot u \cdot b = a^{-1} \cdot c$$

$$u \cdot b \cdot b^{-1} = a^{-1} \cdot c \cdot b^{-1}$$

$$u \cdot e = u = a^{-1} \cdot c \cdot b^{-1}$$

$$\boxed{u = a^{-1} \cdot c \cdot b^{-1}}$$

This solution is unique because a^{-1} & b^{-1} , the inverses of a & b respectively, are unique by definition.

- (d) (2 pts)
- B_5
- (the set of odd permutations in
- S_5
-) is a subgroup of
- S_5
- .

False: $B_5 \not\subseteq S_5$

②

because the identity permutation of S_5 is an even permutation which is not included in B_5 , the set of odd perm. in S_5 .

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- (e) (2 pts)
- B_5
- has 60 elements.

(True).

$$S_5 \text{ has } 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 \text{ elements.}$$

since we could prove (as we did in class) that
there's a bijection ϕ between A_5 and B_5 (2)

recall that it was: $\phi: A_5 \rightarrow B_5$

~~such that $\phi(\sigma) = P \sigma P^{-1} \forall \sigma \in A_5$
where P is a fixed odd perm in B_5 .~~

- (f) (2 pts) If
- a
- generates a group
- G
- of order 12 then
- a^9
- also generates
- G
- .

(False)

Because 9 & 12 are NOT relatively prime (2)
(g.c.d (9, 12) = 3 ≠ 1)

a^9 generates the subgroup of order $= \frac{12}{3} = 4$

$$\langle a^3 \rangle = \langle a^9 \rangle = \left\{ a^9; a^{18} = a^{12} \cdot a^6 = a^6; a^{27} = (a^{12})^2 \cdot a^3 = a^3; a^{12} = e \right\}$$

- (g) (2 pts) There exists an infinite cyclic group having 3 generators.

If a group is infinitely cyclic, then it is isomorphic to $(\mathbb{Z}, +)$. (2)

$(\mathbb{Z}, +)$ is an infinite cyclic group having only exactly two generators $\{1, -1\}$; so any infinite cyclic group must have two generators exactly.

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4. Let $\phi: G \rightarrow K$ be a group homomorphism and let $H = \{g \in G \mid \phi(g) = e_K\}$.

(a) (5 pts) Show that H is a subgroup of G .

- Must prove $H \leq G$

* we have that $H \subseteq G$

$$H = \{g \in G \mid \phi(g) = e_K\}$$

5

i) H is closed under the operation of G :

take $g_1, g_2 \in H \subseteq G$, then $\phi(g_1) = \phi(g_2) = e_K$

must prove that $g_1 g_2 \in H$:

because ϕ is a homomorphism, it holds that

$$\phi(g_1 g_2) = \phi(g_1) \cdot \phi(g_2) = e_K \cdot e_K = e_K$$

so $g_1 g_2 \in H$

ii) let $e_G = \text{id. elt. of } G$

$e_G \in H$ because: ~~$\phi(e_G) = e_K$~~ because ϕ is homomorphism

$$\phi(e_G) = \phi(e_G \cdot e_G) = \phi(e_G) \cdot \phi(e_G)$$

since $\phi(e_G) \in \text{group } K$, then left cancellation law holds:

$$\phi^{-1}(e_G) \cdot \phi(e_G) = \phi^{-1}(e_G) \cdot \phi(e_G) \cdot \phi(e_G)$$

$$\Rightarrow e_K = \phi^{-1}(e_G) \cdot \phi(e_G) = \phi(e_G)$$

$$\Rightarrow \phi(e_G) = e_K \Rightarrow e_G \in H.$$

please see back side →

iii) inverses:

let $h \in H \subseteq G$

then $h^{-1} \in G$

it follows that $h^{-1} \in H$

because:

$$\begin{aligned}\phi(h^{-1}) &= \phi(h^{-1} \cdot h) \\ &= \phi(h^{-1}) \cdot \phi\end{aligned}$$

$$\phi(e_G) = \underbrace{\phi(h \cdot h^{-1})}_{\text{since } \phi \text{ is hom.}} = \phi(h) \cdot \phi(h^{-1})$$

but $\phi(e_G) = \phi(h) = e_K$ since $e_G, h \in H$

$$\text{so } e_K = e_K \cdot \phi(h^{-1})$$

since $\phi(h^{-1}) \in K$ group,

$$\text{then } \cancel{\phi(h^{-1})} e_K \cdot \phi(h^{-1}) = \phi(h^{-1})$$

$$\Rightarrow e_K = \phi(h^{-1})$$

$$\Rightarrow h^{-1} \in H \quad \checkmark$$

hence $H \leq G$.

Name: Ibrahim Fatabi(b) (5 pts) Prove that ϕ is one-to-one if and only if $H = \{e_G\}$.take $\phi(h_1) = \phi(h_2)$ for some $h_1, h_2 \in H \subseteq G$

$$\phi(h_1 h_2) = \phi(h_1) \phi(h_2)$$

①

Suppose ϕ is one-to-one.Assume $H \neq \{e_G\}$ we know that $\phi(e_G) = e_K$
so $e_G \in H$ Let $g \neq e_G$ be an element of H so $\phi(g) = e_K = \phi(e_G)$ impossible since ϕ is 1-1Assume $H = \{e_G\}$. Show ϕ is 1-1.suppose ϕ is not 1-1. So $\exists g_1, g_2 \in G$ such that $\phi(g_1) = \phi(g_2)$

$$\phi(g_1) (\phi(g_2))^{-1} = e_K$$

$$\phi(g_1) \phi(g_2^{-1}) = e_K$$

$$\phi(g_1 g_2^{-1}) = e_K$$

so $g_1 g_2^{-1} \in H \Rightarrow g_1 g_2^{-1} = e_G$
 $g_1 = g_2$, so ϕ is one-to-one.

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5. (5 pts) Let G be a finite group. Show that a nonempty subset H of G is a subgroup of G if and only if $ab \in H$ for all $a, b \in H$.

G is a group.

$$H \subseteq G \quad \text{where } H \neq \emptyset$$

$$\& H \neq \emptyset$$

i) must prove that if $H \leq G$, then $ab \in H \quad \forall a, b \in H$
 this is trivial because if H is a subgroup of G ,
 then it is closed under the operation of G ;
 that is, $\forall a, b \in H \subseteq G, ab \in H$.

ii) must prove that if $\forall a, b \in H, ab \in H$,
 then $H \leq G$.

① closure under operation:

by our supposition, it is true
 that $ab \in H \quad \forall a, b \in H$.

(4)

② identity elt. of G :

let $a \in H \subseteq G$

since $\underbrace{a \cdot e = e \cdot a = a}_{\text{id. elt. of } G} \in H \quad \forall a \in H$,

then e , the id. elt. of G , belongs to H .

③ Inverses:

let $a \in H \subseteq G$; then $a^{-1} \in G \quad \forall a \in G$

and since $a \cdot a^{-1} = a^{-1} \cdot a = e \in H$

$\therefore H$ is a subgroup of G .

(8)